4. Osipov, Iu. S., A differential guidance game for systems with aftereffect. PMM Vol. 35, № 1, 1971.
5. Kurzhanskii, A. B., Differential approach games in systems with lag. Differents. Uravn., Vol. 7, № 8, 1971.
6. Subbotin, A. I., Extremal strategies in differential games with complete memory. Dokl. Akad. Nauk SSSR, Vol. 206, № 3, 1972.

Translated by N. H. C.

UDC 62-50

ON A DIFFERENTIAL-DIFFERENCE GAME OF ESCAPE

PMM Vol. 40, № 6, 1976, pp. 995-1002

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(Received November 10, 1975)

A nonlinear escape problem for conflict-controlled systems described by differential equations with a lagging argument is considered. The sufficient escape conditions which are realized in the class of piecewise-constant functions are obtained. The paper relates to the researches in [1 — 8] and is a continuation of [9, 10].

1. Let a system's motion be described by the differential equation

$$x^{\cdot}(t) = f(x(t), x(t-\tau), u, v), \quad u \in U, \; v \in V \tag{1.1}$$

Here x is the n-dimensional phase vector, u and v are the control parameters of the first and second players, U and V are closed bounded sets. The function $f(x, x_\tau, u, v)$ is continuous in all arguments and is continuously differentiable in x and x_τ, $\tau \geqslant 0$ is the magnitude of the lag. A terminal set M, which is subspace, is delineated in the space E^n. The game terminates if $x(t)$ hits onto set M. As the initial state for the game (1.1) we can take any absolutely continuous function $g(t)$ given in the interval $[-\tau, 0]$. In what follows we assume that derivatives of all the orders needed are present in the functions $g(t)$ used as the intial functions; these derivatives and the functions themselves satisfy in the interval $[-\tau, 0]$ a Lipschitz condition with a constant not exceeding a specified number C.

The vector $x(t)$ moves under the action of the measurable functions $u(t)$ and $v(t)$; the conditions ensuring the continuability of the solution $x(t)$ onto the whole semi-infinite time interval are assumed satisfied. At each instant t the players know the game's state $x_t(s) = x(t+s)$, $-\tau \leqslant s \leqslant 0$. This restricts the information available to the second player from whose position the game is analyzed. We also assume that from the function $x(t)$ specified in some time interval, the escaping can instantly compute its derivatives of all orders needed at any point of the interval.

Let us describe how the game proceeds. From the known current state $x(\cdot)$ (a dot within the parentheses means that the function $x(t)$ on the whole, is being treated as an element of a functional space) the second player determines a number $\varepsilon(x(\cdot)) > 0$, selects a control $v(t) = v(x(\cdot);\; t)$, $0 \leqslant t \leqslant \varepsilon(x(\cdot))$, and informs his opponent. On the basis of the information received the first player sets his own control $u(t)$

in the interval $[0, \varepsilon(x(\cdot))]$. The trajectory $x(t)$ corresponds to the pair of controls selected. The second player's task is to keep the trajectory $x(t)$ from making contact with set M in the whole interval $[0, \infty)$ for any initial function $g(t)$, $g(0) \overline{\in} M$, satisfying the conditions stipulated earlier. If this task is feasible, then we say that escape is possible in game (1.1).

2. By L we denote the orthogonal complement to subspace M in E^n and we assume that $\dim L = \nu \geqslant 2$. Let π be the orthogonal projection operator from E^n onto L. When an initial function $g(t)$ is given in the interval $[-\tau, 0]$ the trajectory $x(t)$ satisfies the ordinary differential equation

$$x^{\cdot} = h(x, t, u, v) \tag{2.1}$$
$$h(x, t, u, v) = f(x, g(t-\tau), u, v)$$

in the interval $[0, \tau]$. We assume the function $f(x, x_\tau, u, v)$ has derivatives of all the orders needed with respect to x and x_τ. By $\nabla_x \varphi(x, \xi)$ we denote the Jacobi matrix for the function $\varphi(x, \xi)$ which is differentiable in x, consisting of the first derivatives of the function with respect to x; we use the recurrence relation

$$\varphi^{(i)}(x, t, u, v) = \nabla_x \varphi^{(i-1)}(x, t, u, v)\, h(x, t, u, v) + \frac{\partial \varphi^{(i-1)}(x, t, u, v)}{\partial t} \tag{2.2}$$

$$\varphi^{(0)}(x, t, u, v) = \pi x, \quad t \in [0, \tau]$$

to form a sequence of functions. As follows from (2.2) the functions $\varphi^{(i)}(x, t, u, v)$ depend upon the initial function $g(t)$ and its derivatives up to order $i-1$. To stress this dependency, we denote $y_i = y_i(t) = (g(t-\tau), g^{(1)}(t-\tau), \ldots, g^{(i-1)}(t-\tau))$ and set

$$\varphi^{(i)}(x, t, u, v) = f^i(x, y_i, u, v), \quad i = 1, 2, \ldots$$

We denote $I = \{1, \ldots, n\}$ and by I_i we denote the collection of indices $j \in I$ for which the function $\varphi^{(i)}(x, t, u, v)$ depends upon x_j, $i = 0, 1, \ldots$. We take system (1.1) to be such that the functions $\varphi^{(i)}(x, t, u, v)$ do not depend upon $x_j(t-\tau)$ for

$$j \in I \setminus \bigcup_{m=0}^{i} I_m$$

Assumption 1. A number k, $k \leqslant \nu - 1$ exists such that the sets $f^i(x, y_i, u, v)$, $i = 1, \ldots, k-1$ consist of unique points.

We set $y_k = y$, $f^i(x, y_i, u, v) = f^i(x, y_i)$, $i = 1, \ldots, k-1$ and for some subspace R of L we denote

$$\Psi_R = \{p\colon p \in R, \|p\| = 1\}$$

Assumption 2. The function $f^k(x, y, u, v)$ depends upon v; there exists a subspace $R \subset L$ and a function $l(x, y)$ continuous in all its arguments, such that

$$F(x, y) = \min_{p \in \Psi_R} \max_{v \in V} \min_{u \in U} (p, f^k(x, y, u, v) - l(x, y)) > 0$$

for all y and $x \in M$, and

$$\dim R \geqslant \max_{x,\, y} \operatorname{rank} B(x, y) \tag{2.3}$$

$$B(x, y) = \begin{pmatrix} \pi x \\ f^1(x, y_1) \\ \cdots\cdots \\ f^{k-1}(x, y_{k-1}) \\ l(x, y) \end{pmatrix}$$

Theorem. If Assumptions 1 and 2 are satisfied, then an escape, realized in the class of piecewise-constant functions $v(t)$, is possible in the differential-difference game (1.1).

3. To prove the theorem we consider the many-valued mapping

$$S(y) = \{x: F(x, y) \geqslant 0\}$$

By Assumption 2, $S(y) \neq \varnothing$ for any y. In addition, the mapping $S(y)$ is closed and, because function $F(x, y)$ is continuous, any point $x \in M$ belongs to $\operatorname{int} S(y)$. Let an initial function $g_0(t),\ -\tau \leqslant t \leqslant 0$ be given, and

$$g_0(0) \overline{\in} M,\quad g_0(0) \in \operatorname{int} S(y^0),\quad y^0 = (g_0(-\tau),\ g_0^{(1)}(-\tau), \ldots, g_0^{(k-1)}(-\tau))$$

We fix an element v_0 of V, satisfying the condition

$$\max_{v\in V} \min_{u\in U} (p_0, f^k(g_0(0), y^\circ, u, v) = \min_{u\in U} (p_0, f^k(g_0(0), y^\circ, u, v_0)) \qquad (3.1)$$

where vector p_0 belongs to Ψ_R and satisfies the system of linear inequalities

$$(p_0, g_0(0)) \geqslant 0,\ (p_0, l(g_0(0), y^0)) \geqslant 0,\ (p_0, f^i(g_0(0), y_i^0)) \geqslant 0 \qquad (3.2)$$
$$i = 1, \ldots k-1$$

Because of (2.3) system (3.2) is solvable relative to p_0. Since $g_0(0) \in \operatorname{int} S(y^\circ)$, then $F(g_0(0), y^\circ) > 0$ and all the more

$$\min_{u\in U} (p_0, f^k(g_0(0), y^\circ, u, v_0) - l(g_0(0), y^\circ)) > 0$$

We select a neighborhood of the point $(g_0(0), y^\circ)$ of radius r so small that the inequality

$$\min_{u\in U} (p_0, f^k(x, y, u, v_0) - l(g_0, (0), y^\circ)) \geqslant 0 \qquad (3.3)$$

is satisfied by continuity. We fix a neighborhood of radius $r/2$ for each of the points $g_0(0)$ and $g_0^{(i)}(-\tau),\ i = 1, \ldots, k-1$.

From the assumptions on the sets U and V and on the function $f(x, x_\tau, u, v)$ the use of Gronwall's lemma [11] yields the existence of $t_1,\ 0 < t_1 \leqslant \tau$, such that the trajectory of system (2.1), starting at the point $g_0(0)$ with arbitrary measurable control $u(t)$ and with $v(t) = v_0$, does not leave the neighborhood of radius $r/2$ of point $g_0(0)$ during time t_1. Further, since the function $g_0(t)$ together with its derivatives satisfy a Lipschitz condition with constant C in the interval $[-\tau, 0]$, they lie in neighborhoods of radius $r/2$ of the points $g_0(-\tau)$ and $g_0^{(i)}(-\tau),\ i = 1, \ldots, k-1$ in the interval $[-\tau, r/2C - \tau]$. We set $\varepsilon(g_0(\cdot)) = \min\{t_1, r/2C\}$ and we select the escape control as follows: $v(g_0(\cdot), t) = v_0,\ 0 \leqslant t \leqslant \varepsilon(g_0(\cdot))$. Then with an arbitrary measurable function $u(t)$ we obtain trajectory $x(t)$ by intergrating system (2.1) in the interval $[0, \varepsilon(g_0(\cdot))]$.

Let an initaial function $g_0(t),\ -\tau \leqslant t \leqslant 0$ be given such that $g_0(0) \overline{\in} \operatorname{int} S(y^\circ)$.

We select a neighborhood of radius r of point $g_0(0)$ so that it does not intersect set M. Then $t_2 > 0$ exists such that the trajectory of system (2.1), starting at point $g_0(0)$ with arbitrary measurable controls $u(t) \in U$ and $v(t) \in V$, does not leave the neighborhood of radius r of point $g_0(0)$ during time t_2. We set $\varepsilon(g_0(\cdot)) = t_2$ and the escape control $v(g_0(\cdot), t) = v_0$, $0 \leqslant t \leqslant \varepsilon(g_0(\cdot))$, where $v_0 \in V$ but does not necessarily satisfy (3.1).

Using the stationarity of Eq.(1.1) at the instant $\theta > 0$, we can take $g(s) = x_\theta(s)$, $-\tau \leqslant s \leqslant 0$ as the initial function. From the assumptions on system (1.1) it follows that in the interval $[-\tau, 0]$ the function $g(s)$ together with its derivatives satisfies a Lipschitz condition with a constant depending on the right-hand side of (1.1) and on sets U and V. Thus, the escape control can be constructed by the method described above.

Let us consider the projection, onto the direction of p_0, of the obtained trajectory $x(t)$, $0 \leqslant t \leqslant \varepsilon(g_0(\cdot))$, corresponding to the initial function $g_0(t)$, $g_0(0) \in \operatorname{int} S(y^\circ)$, and to the controls selected. Using the permutability of the operations of projection and differentiation, in accord with Assumption 1 and Taylor's formula [12] we obtain

$$(p_0, x(t)) = (p_0, g_0(0)) + \sum_{i=1}^{k-1} \frac{t^i}{i!}(p_0, f^i(g_0(0), y_i^\circ)) + \int_0^t \frac{(t-\xi)^k}{k!}(p_0, f^k(x(\xi), y(\xi), u(\xi), v_0))\, d\xi \tag{3.4}$$

In its own turn the projection of the remainder term onto the direction of p_0 can be represented in the form

$$\int_0^t \frac{(t-\xi)^k}{k!}(p_0, f^k(x(\xi), y(\xi), u(\xi), v_0))\, d\xi = \int_0^t \frac{(t-\xi)^k}{k!}(p_0, f^k(x(\xi), y(\xi), u(\xi), v_0) - l(g_0(0), y^\circ))\, d\xi + \frac{t^{k+1}}{k+1}(p_0, l(g_0(0), y^\circ)) \tag{3.5}$$

From formula (3.4), with due regard to (3.5), (3.2) and (3.3), follows

$$(p_0, x(t)) > 0, \quad 0 < t \leqslant \varepsilon(g_0(\cdot)) \tag{3.6}$$

We denote the graph of the many-valued mapping $S(y)$ by

$$\operatorname{graph} S = \{(x, y),\ x \in S(y)\}$$

Let us show that for any state $g(\cdot)$ such that the pair $(g(0), y^\circ) \in Z$, where Z is some compactum from the set $\operatorname{graph} S$, we can select the quantity $\varepsilon(g(\cdot)) \geqslant \varepsilon_Z > 0$, where the constant ε_Z depends only on set Z. By Assumption 2

$$\min_{(x, y) \in Z + \delta\Omega} F(x, y) = \Delta > 0$$

Here δ is an arbitrary positive number which is fixed in what follows and Ω is the unit sphere. We consider the function

$$\psi(p, x, y, u, v) = (p, f^k(x, y, u, v) - l(x, y))$$

It is uniformly continuous in all its arguments on the set $\Psi_R \times (Z + \delta\Omega) \times U \times V$.

Consequently, $\eta = \eta(\Delta) > 0$, $\eta \leqslant \delta$ exists such that in a neighborhood of any point $(x_0, y^\circ) \in Z$ of radius not less than η the function

$$\min_{u \in U} (p_0, f^k(x, y, u, v_0) - l(x_0, y^\circ))$$

where p_0 and v_0 are computed with respect to point (x_0, y°) by the relations (3.1) and (3.2), remains nonnegative. Since $x(t)$ and $y(t)$ are absolutely continuous functions, the function $z(t) = (x(t), y(t))$ satisfies in the region $Z + \delta\Omega$ a Lipschitz condition with a constant K depending on the set $Z + \delta\Omega$. Then the function $z(t)$, starting from any initial point $z_0 = (x_0, y_0) \in Z$, does not leave the region $Z + \delta\Omega$ during time η / K and, consequently, we can choose

$$\varepsilon(g(\cdot)) = \varepsilon(z) \geqslant h / K > 0$$

for all $z \in Z$. From the method of constructing the escape strategy and from the result obtained it follows that $\varepsilon(z) \geqslant \varepsilon_Z > 0$ for any compact set Z.

Thus, if θ' and θ'' are instants such that $\theta'' = \theta' + \varepsilon(x_{\theta'}(\cdot))$, then the trajectory $x(t)$ does not intersect set M in the interval $[\theta', \theta'']$. We construct the escape control from the known current state by the method indicated earlier. Let us prove that trajectory $x(t)$ does not intersect M in the interval $[0, \infty)$. Indeed, let $T > 0$ be any finite time. By the assumptions on the parameters of game (1.1) the curve $z(t)$ has not left some compact set Z by this time under any controls. Since we can choose $\varepsilon(z) \geqslant \varepsilon_Z > 0$ for $z \in Z$, no more than $[T / \varepsilon_Z]$ changes of control take place in time T, i.e. there are only a finite number of instants $t_1, t_2, \ldots, t_m$ at which the choice of escape control changed. In each of the intervals $[t_i, t_{i+1}]$ the control $v(t)$ was chosen such that $x(t) \overline{\in} M$. Consequently, $x(t) \overline{\in} M$ in the whole interval $[0, T]$, which completes the proof since T was chosen arbitrarily.

Note 1. We relax the requirement $F(x, y) > 0$ in Assumption 2, replacing it by:

$$G(x, y) = \min_{p \in \Psi_R} \min_{u \in U} \max_{v \in V} (p, f^k(x, y, u, v) - l(x, y)) > 0$$

Then the theorem's proof holds with the sole difference that escape is realized in the class of measurable functions $v(t)$ with the use of information at instant t both on the current position as well as on the control $u(t)$ [5 — 8]. The proof of this fact is analogous to that of the theorem and uses the result in [13].

Note 2. Inequality (2.3) can be replaced by the simpler one $\dim R \geqslant k + 1$, since

$$\max_{x, y} \operatorname{rank} B(x, y) \leqslant k + 1$$

4. From the theorem and Note 1 follow a number of corollaries suitable for analyzing escape possibilities and for constructing the escape control. We carry out the proof only for the cases not assuming information discrimination against the pursuer. The proofs of the results in the other case are similar. Let the right-hand side of system (1.1) have the form

$$f(x(t), x(t - \tau), u, v) = f_1(x(t), x(t - \tau)) + f_2(u, v)$$

Corollary 1. Let a two-dimensional subspace R, $R \subset L$ and a vector l exist such that

$$\min_{p \in \Psi_R} \max_{v \in V} \min_{u \in U} (p, f_2(u, v) - l) > 0 \qquad (4.1)$$

$$\left(\min_{p \in \Psi_R} \min_{u \in U} \max_{v \in V} (p, f_2(u, v) - l) > 0\right)$$

Then escape is possible in game (1. 1), realized in the class of piecewise-constant (measurable) functions $v(t)$.

Proof. Let the initial function $g_0(t), -\tau \leqslant t \leqslant 0$, be given, where $g_0(0) \in M$. We choose the element v_0 of V from the condition

$$\max_{v \in V} \min_{u \in U} (p_0, f_2(u, v)) = \min_{u \in U} (p_0, f_2(u, v_0)) \tag{4.2}$$

where $p_0 \in \Psi_R$ and satisfies the system of linear inequalities

$$(p_0, g_0(0)) \geqslant 0, \quad (p_0, f_1 (g_0(0), g_0(-\tau)) + l) \geqslant 0 \tag{4.3}$$

Then

$$(p_0, f_1(g_0(0), g_0(-\tau)) + \min_{u \in U} (p_0, f_2(u, v_0)) = (p_0, f_1(g_0(0), g_0(-\tau)) + l) + \min_{u \in U} (p_0, f_2(u, v_0) - l) > 0$$

because of (4. 1) — (4. 3). We select a neighborhood of the point $(g_0(0), g_0(-\tau))$ of radius r so small that the inequality

$$(p_0, f_1(x, y)) + \min_{u \in U} (p_0, f_2(u, v_0)) \geqslant 0 \tag{4.4}$$

is satisfied by continuity. Having chosen a neighborhood of radius $r/2$ of each of the point $g_0(0)$ and $g_0(-\tau)$, we construct the escape control just as we did in the theorem's proof. As a result of integrating system (1. 2) with a chosen control $u(t)$ we obtain trajectory $x(t)$ in the interval $[0, \varepsilon(g_0(\cdot))]$, $\varepsilon(g_0(\cdot)) \leqslant \tau$. From formula (3. 4), with due regard to (4. 3) and (4. 4), we obtain an estimate for the projections of the trajectory obtained onto the direction of p_0

$$(p_0, x(t)) > 0, \quad 0 < t \leqslant \varepsilon(g_0(\cdot))$$

Let us show that for any state $g(\cdot)$ such that the pair $(g(0), g(-\tau))$ belongs to a compactum Z from E^{2n}, we can choose $\varepsilon(g(\cdot)) \geqslant \varepsilon_Z > 0$, where the constant ε_Z depends only on set Z. Because of the way p_0 was chosen, at any point (x_0, y°) we have $(p_0, f_1(x_0, y^\circ)) + \min_{u \in U} (p_0, f_2(u, v_0)) \geqslant \Delta > 0$

$$\Delta = \min_{p \in \Psi_R} \max_{v \in V} \min_{u \in U} (p, f_2(u, v) - l)$$

the function $f_1(x, y)$ is continuous in all its arguments. Then a number $\eta = \eta(\Delta) > 0$ exists such that in a neighborhood of the point $(x_0, y^\circ) \in Z$ of radius not less than η, the function in the left-hand side of inequality (4. 4) wherein the vectors p_0 and v_0 have been computed from the point (x_0, y°) by (4. 2) and (4. 3) remains nonnegative.

Since $x(t)$ is an absolutely continuous function, the function $z(t) = (x(t), x(t-\tau))$ is absolutely continuous and, consequently, satisfies in region $Z + \eta\Omega$ a Lipschitz condition with a constant K depending on set $Z + \eta\Omega$. The function $z(t)$ starting from any point $z_0 = (x_0, y^\circ) \in Z$ does not leave the region $Z + \eta\Omega$ during time η / K and, consequently, we can select

$$\varepsilon(g(\cdot)) = \varepsilon(z) \geqslant \eta / k > 0$$

for all $z \in Z$. The concluding part of the proof repeats the corresponding arguments from the theorem's proof.

Let the right-hand side of system (1. 1) have the form

$$f(x(t), x(t-\tau), u, v) = A_1 x(t) + A_2 (x) (t-\tau) + f_2(u, v)$$

A_1 and A_2 are ($n \times n$)-matrices.

Corollary 2. Let there exist a number k, a subspace R, $R \subset L$, $R \geqslant k+1$ and a vector l such that the sets $\pi A_1^i f_2(U, V)$, $i = 0, \ldots, k-2$ consist of unique points and

$$(\min_{p\in\Psi_R}\max_{v\in V}\min_{u\in U}(p,\ A_1^{k-1}f_2(u,v)-l)>0 \tag{4.5}$$

$$(\min_{p\in\Psi_R}\min_{u\in U}\max_{v\in V}(p,\ A_1^{k-1}f_2(u,v)-l)>0)$$

Then escape is possible in game (1.1), realized in the class of piecewise-constant (measurable) functions $v(t)$.

Proof. Let the initial function $g_0(t)$, $-\tau \leqslant t < 0$ be given. We select the element v_0 of V from the condition $\max\limits_{v\in V}\min\limits_{u\in U}(p_0,\ A_1^{k-1}f_2(u,v)) = \min\limits_{u\in U}(p_0,\ A_1^{k-1}f_2(u,v_0))$ (4.6) where $p_0 \in \Psi_R$ and satisfies the system of linear inequalities

$$\begin{gathered}(p_0,\ g_0(0))\geqslant 0\\ (p_0,\ A_1^{i+1}g_0(0)+\sum_{j=0}^{i}A_1^{i-j}A_2g_0^{(j)}(-\tau)+A_1^{i}f_2(u,v))\geqslant 0,\quad i=0,\ldots,k-2\\ (p_0,\ A_1^{k}g_0(0)+\sum_{j=0}^{k-1}A_1^{k-j-1}A_2g_0^{(j)}(-\tau)+l)\geqslant 0\end{gathered} \tag{4.7}$$

Here the $A_1^{i}f_2(u,v)$ are certain vectors since the first hypothesis of Corollary 1 is satisfied. Since $k+1\leqslant \nu$, system (4.7) is solvable relative to p_0. Then

$$\begin{gathered}\bigl(p_0,\ A_1^{k}g_0(0)+\sum_{j=0}^{k-1}A_1^{k-j-1}A_2g_0^{(j)}(-\tau)+\min_{u\in U}(p_0,\ A_1^{k-1}f_2(u,v_0))=\\ (p_0,A_1^{k}g_0(0))+(p_0,\sum_{j=0}^{k-1}A_1^{k-j-1}A_2g_0^{(j)}(-\tau)+l)+\min_{u\in U}(p_0,\ A_1^{k-1}f_2(u,v_0)-l)>0\end{gathered} \tag{4.8}$$

because of (4.5)—(4.7).

The length $\varepsilon(g_0(\cdot))$ of the interval of constancy for the escape control v_0 is chosen just as in the theorem's proof with the use of (4.8). Estimate (3.6) can be obtained similarly. For any state $g(\cdot)$ such that the set $(g(\cdot), g(-\tau), g^{(1)}(-\tau), \ldots, g^{(k-1)}(-\tau))$ belongs to compactum Z of $E^{n(k+1)}$, we can select $\varepsilon(g(\cdot))\geqslant\varepsilon_Z>0$; this fact is proved similarly to the proof of the theorem and of Corollary 1. The concluding part repeats the theorem's proof.

REFERENCES

1. Krasovskii, N. N. and Subbotin, A. I., Position Differential Games. Moscow, "Nauka", 1974.
2. Krasovskii, N. N. and Osipov, Iu. S., Linear differential-difference games. Dokl. Akad. Nauk SSSR, Vol. 197, № 4, 1971.
3. Osipov, Iu. S., Alternative in a differential-difference game. Dokl. Akad. Nauk SSSR, Vol. 197, № 5, 1971.
4. Osipov, Iu. S., Minimax absorption in differential-difference games. Dokl. Akad. Nauk SSSR, Vol. 203, № 1, 1972.
5. Pontriagin, L. S. and Mishchenko, E. F., Problem of evasion of contact in linear differential games. Differents. Uravnen., Vol. 7, № 3, 1971.
6. Pshenichnyi, B. N., On the escape problem. Kibernetika, № 4, 1975.
7. Nikol'skii, M. S., On the linear escape problem. Dokl. Akad. Nauk SSSR, Vol. 218, № 5, 1974.

8. Nikol'skii, M. S., Linear differential evasion game in the presence of lags. In: Applied Mathematics and Programing, Issue 10, Kishinev, "Shtiintsa", 1973.
9. Chikrii, A. A., The evasion problem in nonlinear differential games. Kibernetika, №3, 1975.
10. Chikrii, A. A., The evasion problem in nonstationary differential games. PMM Vol. 39, №5, 1975.
11. Sansone, G., Ordinary Differential Equations. Vol. 1, Moscow, Izd. Inostr. Lit., 1953.
12. Kartan, A., Differential Calculus. Moscow, "Mir", 1971.
13. Chikrii, A. A. On a class of nonlinear evasion games. Kibernetika, №3, 1976.

Translated by N. H. C.

UDC 62-50

THE GAME PROBLEM ON THE DOLICHOBRACHISTOCHRONE

PMM Vol. 40, №6, 1976, pp. 1003-1013
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(Received November 11, 1975)

The capture and evasion sets, the players' optimal strategies and the game value determined for the game problem on the dolichobrachistochrone, analysed within the framework of a position formalism similar to [1]. Singularities inherent in the game of the minimax-maximin time to contact [1, 2] become apparent; they are determined in the given problem by the specific behavior of the optimal paths close to the target set. Isaacs [4] examined the game problem on the dolichobrachistochrone, being the game analog of the classical variational problem on the brachistochrone [3]. However, as was shown in [5], the solution proposed by Isaacs contains erroneous statements.

1. In the game problem on the dolichobrachistochrone a point m moves in the half-plane of x and y $(y \geqslant 0)$ in accord with the equation

$$x^{\cdot} = \sqrt{y} \cos u + w (v + 1) / 2, \quad y^{\cdot} = \sqrt{y} \sin u + w (v - 1) / 2 \tag{1.1}$$

Here w is a positive constant and u and v are control parameters subject to the first and second players, respectively, and to the constraints

$$0 \leqslant u \leqslant 2\pi, \quad -1 \leqslant v \leqslant 1 \tag{1.2}$$

The first player's aim is the most rapid approach of point m the target set

$$M = \{p = \{x, y\} \mid x = 0, \ y \geqslant 0\} \tag{1.3}$$

being positive part of the ordinate axis. The second player tries to prevent point m from hitting onto set M or, at least, to delay it. In the problem statement we assume that point m is in the first quadrant at the initial instant.

In [4] it is stated that for initial points x_0 and y_0 satisfying the conditions $x_0 > 0$ and $0 \leqslant y_0 < w^2$ the second player can prevent approach to the target set M in spite of any efforts of the first player. This statement is justified in [4] in the following manner: in